RESTRICTED

Copy No. 59

A M P Note No. 15
AMG-NYU No. 53

COMPLEX ROOTS OF POLYNOMIAL EQUATIONS

by B. Friedman

WITH THE APPROVAL OF THE OFFICE OF THE CHAIRMAN OF THE NATIONAL DEFENSE RESEARCH COMMITTEE, THIS REPORT HAS BEEN DECLASSIFIED BY THE OFFICE OF SCIENTIFIC RESEARCH AND DEVELOPMENT.

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

Prepared For The
APPLIED MATHEMATICS PANEL
NATIONAL DEFENSE RESEARCH COMMITTEE
By The
Applied Mathematics Group
New York University

THIS DOCUMENT CONTAINS INFORMATION AFFECTING THE NATIONAL DEFENSE OF THE UNITED STATES WITHIN THE MEANING OF THE ESPIONAGE ACT, U. S. C. 50; 31 AND 32. ITS TRANSMISSION OR THE REVELATION OF ITS CONTENTS IN ANY MANNER TO AN UNAUTHORIZED PERSON IS PROHIBITED BY LAW.

August 1944

RESTRICTED

Errata for AMP Note No. 15

Two minor computational errors which do not affect the validity of the report have been found in this Note, as follows:

Page 2, third line from bottom:

Instead of: $7.581x^2 + 4.053x^3 + x^4$

Read: $7.481x^2 + 4.053x^3 + x^4$

Page 10, fourth line from top:

Instead of: $x^2 + .725x + 1.98$

Read: $x^2 + .725x + 1.99$

DISTRIBUTION LIST
AMP Note No. 15

No. of Copies	
2	Office of the Executive Secretary, OSRD
15	Liaison Office, OSRD
	1 Admiralty Undex Works
	3 Air Defense Research and Development Establishment
	A. Porter
	T. Pearcey
	C. Mack
	1 Admiralty Computing Service
	1 Dr. L. J. Comrie
1	War Department Liaison Officer with NDRC
	Att: H. E. Clements
1	Co-ordinator of Research and Development, Navy
	Att: Lt. J. H. Wakelin
1	Ordnance Dept., Tech. Service Division
	H. M. Morse
1	Ordnance Research Center, Aberdeen Proving Ground
	O. Veblen
3	J. T. Tate, Chief, Division 6
	1 C. L. Pekeris
	1 Underwater Sound Laboratory, Harvard
2	A. L. Loomis, Chief, Division 14
	1 P. D. Crout
1	Warren Weaver, Chief, AMP
1	T. C. Fry
6	R. Courant
1	S. S. Wilks
1	M. Rees
1	I. S. Sokolnikoff
2	L. J. Briggs
	1 Att: A. N. Lowan
1	J. G. Kirkwood
1	S. MacLane
1	P. M. Morse
1	E. J. Moulton
1	W. Prager
1	A. H. Taub
1	J. Schilt
1	J. von Neumann
1	W. A. Wallis

INTRODUCTION

While the present note does not originate from a specific request, its publication by the Applied Mathematics Panel seems justified since it concerns the problem of determining numerically the complex roots of algebraic equations. Within the NDRC and in related British agencies, the need for such numerical methods has become apparent; hence the following memorandum by Dr. B. Friedman of the New York University Applied Mathematics Group is being issued by the Applied Mathematics Panel. This note is an amplification of a short abstract presented to the American Mathematical Society (Bulletin of American Mathematical Society, v. 49, page 859, 1943).

R. Courant
Contractor's Technical
Representative

COMPLEX ROOTS OF POLYNOMIAL EQUATIONS

The problem of finding the complex zeros of a polynomial equation is important in many branches of science and engineering. Graeffe's method, which is very powerful when the roots are real, is not practical when most of the roots are complex.

Recently a British report* described a method for calculating the roots of equations of the fifth and sixth degree. One disadvantage of the proposed method is that, at each step, it requires the solution of a quadratic equation.

This report presents a simple, general method for finding the complex roots of any polynomial equation. All the operations can be easily performed on a calculating machine.

Since complex numbers are awkward to handle, it is desirable to operate with the quadratic factors of the polynomial instead of with its complex roots. The question of finding quadratic factors can be divided into two problems: First, to obtain quickly an approximate value for the quadratic factor and second, to improve the approximation. The first problem will be answered by a method of successive

---

* WA-1907-1 A.D.R.D.E. Res. Rep. No. 238

forward and back divisions;* the second, by an extension of Newton's method to complex roots.

Method I. Successive Division

The method can be best described by considering a particular example:

Find the roots of

$$f(x) = x^4 + 5x^3 + 13x^2 + 16x + 10 = (x^2 + 2x + 2)(x^2 + 3x + 5)$$

Assume as a trial factor $x^2 + 0x + .25$. (A method for determining the first trial factor will be given later.) Then

$$\frac{f(x)}{x^2 + .25} = x^2 + 5x + 12.75 + \frac{14.75x + 6.8125}{x^2 + .25} .$$

Divide $f(x)$ by $12.75 + 5x + x^2$ in ascending powers of x, as follows:

$$
\begin{array}{r|lllll}
\multicolumn{1}{r}{} & .784 & + .947x & + .595x^2 & & \\ \cline{2-6}
12.75 + 5x + x^2 & 10 & + 16x & + 13x^2 & + 5x^3 & + x^4 \\
\multicolumn{1}{r}{} & 10 & + 3.92x & + .784x^2 & & \\ \cline{2-4}
\multicolumn{1}{r}{} & & 12.08x & + 12.216x^2 & + 5x^3 & \\
\multicolumn{1}{r}{} & & 12.08x & + 4.735x^2 & + .947x^3 & \\ \cline{3-5}
\multicolumn{1}{r}{} & & & 7.581x^2 & + 4.053x^3 & + x^4 \\
\multicolumn{1}{r}{} & & & 7.581x^2 & + 2.973x^3 & + .595x^4 \\ \cline{4-6}
\multicolumn{1}{r}{} & & & & 1.180x^3 & + .405x^4
\end{array}
$$

---

* This is an extension of a method proposed by Lin, Journal of Mathematics and Physics, June 1943.

The quotient is $.595x^2 + .947x + .784$ or $.595\,(x^2 + 1.59x + 1.32)$, so that $x^2 + 1.59x + 1.32$ is a closer approximation to the factor $x^2 + 2x + 2$.

The motivation for this method can be seen from the following facts:

Suppose the correct factor $x^2 + 2x + 2$ were known. Then

$$\frac{f(x)}{x^2 + 2x + 2} = x^2 + 3x + 5$$

and if $f(x)$ were divided by $5 + 3x + x^2$ (in ascending powers of x), the quotient would be $x^2 + 2x + 2$, the correct factor. This suggests that an iteration method, which consists in dividing $f(x)$ by trial factors in descending and ascending powers of x, successively, may converge to the correct factor.

The method does converge for the example given previously. After five repetitions of the process, the result is $x^2 + 1.98x + 1.99$. By further repetitions, any desired accuracy can be obtained.

The work necessary for this method can be simplified if synthetic division is used. Suppose

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$$

and the trial factor is $x^2 + px + q$. Use only the

coefficients and perform the division as follows:

$$\begin{array}{r|cccccccccccc}
1 & 1 & + & a_1 & + & a_2 & + & a_3 & + & a_4 & + \cdots & + & a_n \\
-p & & - & p & - & b_1p & - & b_2p & - & b_3p & & & \\
-q & & & & - & q & - & b_1q & - & b_2q & - \cdots & - & b_{n-2}q \\
\hline
 & 1 & + & b_1 & + & b_2 & + & b_3 & + & b_4 & + \cdots & + & (a_n - b_{n-2}q)
\end{array}$$

where

$$b_1 = a_1 - p$$

$$b_2 = a_2 - b_1p - q$$

$$b_3 = a_3 - b_2p - b_1q$$

$$b_4 = a_4 - b_3p - b_2q \text{ etc.}$$

Notice that the signs of the coefficients of the quotient have been changed. This permits us to add instead of subtracting. The numbers in the successive diagonals are obtained by multiplying the coefficients of the quotient by the coefficients of the divisor.

As further illustration, suppose we try the next step in this method for the example mentioned previously, that is, divide $f(x)$ by $x^2 + 1.59x + 1.32$.

$$\begin{array}{r|ccccccccc}
1 & 1 & +5 & + & 13 & + & 16 & + & 10 \\
-1.59 & & -1.59 & - & 5.422 & - & 9.95 & & \\
-1.32 & & & - & 1.32 & - & 4.50 & - & 8.26 \\
\hline
 & 1 & +3.41 & + & 6.258 & + & 1.55 & + & 1.14
\end{array}$$

The quotient is $x^2 + 3.41x + 6.258$ while the remainder is $1.55x + 1.14$. The next step would be to divide $10 + 16x + 13x^2 + 5x^3 + x^4$ by $6.258 + 3.41x + x^2$. The work is simplified if the leading coefficient of the divisor is one. Write $6.258 + 3.41x + x^2$ as 6.258 times $(1 + .54x + .16x^2)$. The division proceeds as follows:

$$\begin{array}{r|rrrrr} 1 & 10 & +\ 16 & +\ 13 & +\ 5 & +\ 1 \\ -\ .54 & & -\ 5.4 & -\ 5.7 & & \\ -\ .16 & & & -\ 1.6 & & \\ \hline & 10 & +\ 10.6 & +\ 5.7 & & \end{array}$$

Since the remainder is not needed, the work can stop here. Now the quotient is $10 + 10.6x + 5.7x^2 = 5.7(x^2 + 1.86x + 1.75)$ and $x^2 + 1.86x + 1.75$ is the approximation to the correct factor.

The whole process can be combined as follows:*

$$\begin{array}{r|rrr} 1 & 1\ +\ 5 & & +\ 13 \\ -\ 1.59 & & -\ 1.59 & -\ 5.422 \\ -\ 1.32 & & & -\ 1.32 \\ \hline & 1\ +\ 3.41 & & +\ 6.258 \end{array} \qquad \begin{array}{rrr|r} 13 & +\ 16 & +\ 10 & 1 \\ -\ 5.7 & -\ 5.4 & & -\ .54 \\ -\ 1.6 & & & -\ .16 \\ \hline +\ 5.7 & +\ 10.6 & +\ 10 & \end{array}$$

Note that $6.258(1 + \frac{3.41}{6.258}x + \frac{1}{6.258}x^2) = 6.258(1 + .54x + .16x^2)$

The next approximation is $x^2 + \frac{10.6}{5.7}x + \frac{10}{5.7}$.

Two more repetitions of this forward and back division give as the factor $x^2 + 1.98x + 1.99$. The process can be continued to give any needed accuracy.

---

* The preceding synthetic division is written in the reverse direction merely for convenience of computation.

This method can be applied to polynomials of any degree. In general it will converge to the quadratic factor containing the two roots of smallest absolute value. The initial trial factor should be chosen so that its roots are smaller in absolute value than those of the given equation.

Simple reasoning gives a lower bound for the absolute value of the roots. Consider the equation given in the British report referred to on page 1:

$$g(x) = x^6 + 14.71x^5 + 92.76x^4 + 273.3x^3 + 515x^2 + 532x + 435.5 = 0 \; .$$

The following inequality is obtained:

$$\begin{aligned} &|x^6 + 14.71x^5 + 92.76x^4 + 273.3x^3 + 515x^2 + 532x + 435.5| \\ &\quad \geq 435.5 - 532|x| - 515|x^2| - 273.3|x^3| - 92.76|x^4| \\ &\quad - 14.71|x^5| - |x|^6 \end{aligned}$$

The right hand side is greater than zero if $|x| > .77$. A trial factor can be then taken as $x^2 + (.77)^2$ or $x^2 + .6$.

The work proceeds as follows:

$$\begin{array}{r|ccccccccc|ccccc|l}
1 & 1 & + & 14.71 & + & 92.76 & + & 273.3 & + & 515 & 515 & + & 532 & + & 435.5 & 1 \\
0 & & & 0 & & 0 & + & 0 & & 0 & -162 & - & 248 & & & -.57 \\
-.6 & & & & - & .6 & - & 8.8 & - & 55.3 & -87 & & & & & -.2 \\
\hline
 & 1 & + & 14.71 & + & 92.16 & + & 264.5 & + & 459.7 & 266 & + & 284 & + & 435.5 &
\end{array}$$

Note that the "back" divisor is $1 + \frac{264.5}{459.7}x + \frac{92.16}{459.7}x^2$

$$= 1 + .57x + .2x^2 \; .$$

The second approximation is $x^2 + \frac{284}{266}x + \frac{435.5}{266}$

$= x^2 + 1.1x + 1.6$

$$\begin{array}{r|rrrrrrrr|l}
1 & 1 & +\,14.71 & +\,92.76 & +\,273.3 & +\,515 & 515 & +\,532 & +\,435.5 & 1 \\
-1.1 & & -\,1.1 & -\,14.97 & -\,83.8 & -\,184.4 & -146.9 & -\,348.4 & & -\,.80 \\
-1.6 & & & -\,1.6 & -\,21.8 & -\,121.9 & -161.1 & & & -\,.37 \\ \hline
& 1 & +\,13.61 & +\,76.2 & +\,167.7 & +\,208.7 & 207 & +\,183.6 & +\,435.5 &
\end{array}$$

The third approximation is $x^2 + \frac{183.6x}{207} + \frac{435.5}{207}$

$= x^2 + .89x + 2.1$.

$$\begin{array}{r|rrrrrrrr|l}
1 & 1 & +\,14.71 & +\,92.76 & +\,273.3 & +\,515 & 515 & +\,532 & +\,435.5 & 1 \\
-\,.89 & & -\,.89 & -\,12.30 & -\,69.7 & -\,155.4 & -126 & -\,392 & & -\,.90 \\
-2.1 & & & -\,2.1 & -\,29.0 & -\,164.6 & -174.2 & & & -\,.40 \\ \hline
& 1 & +\,13.82 & +\,78.36 & +\,174.6 & +\,195.0 & 214.8 & +\,140 & +\,435.5 &
\end{array}$$

The fourth approximation is $x^2 + .65x + 2.03$.

This method can be continued as long as necessary, but, since the approximation is fairly close to the correct factor, another method which converges more quickly can be used.

Method II. Generalized Newton's Method*

Suppose $f(x)$ is divided by some trial factor $x^2 + px + q$. A quotient $Q(x)$ and a remainder $rx + s$ will be obtained:

---

* A somewhat similar method has been proposed independently by Hitchcock, Journal of Mathematics and Physics, May 1944.

(1) $f(x) = (x^2 + px + q)\ Q(x) + rx + s$, where r and s are functions of p and q. The problem is to change p and q so that r and s will be zero. Using Taylor's theorem in two variables, this can be done if the partial derivatives of r and s with respect to p and q are known.

Differentiate equation (1) with respect to p. The result is

$$(3)\quad 0 = xQ(x) + (x^2 + px + q)\frac{\partial Q}{\partial p} + \frac{\partial r}{\partial p}x + \frac{\partial s}{\partial p}$$

If the zeros of $x^2 + px + q$ are substituted for x, it is seen that

$\frac{\partial r}{\partial p}x + \frac{\partial s}{\partial p}$ = remainder of $[-xQ(x)]$ when divided by $x^2 + px + q$.

Similarly

$\frac{\partial r}{\partial q}x + \frac{\partial r}{\partial q}$ = remainder of $[-Q(x)]$ when divided by $x^2 + px + q$.

Using Taylor's theorem, the solution of the equations*

$$(3)\quad \begin{aligned} \frac{\partial r}{\partial p}\Delta p + \frac{\partial r}{\partial q}\Delta q &= -r \\ \frac{\partial s}{\partial p}\Delta p + \frac{\partial s}{\partial q}\Delta q &= -s \end{aligned}$$

will give the changes in p and q which will cause r and s to become zero.

As an illustration consider the previous example.

$$f(x) = x^6 + 14.71x^5 + 92.76x^4 + 273.3x^3 + 515x^2 + 532x + 435.5$$

where $x^2 + px + q = x^2 + .65x + 2.03$

---

* for Δp and Δq.

$$\begin{array}{r|rrrrrrr}
1 & 1 & +14.71 & +92.76 & +273.3 & +515 & +532 & +435.5 \\
-.65 & & -.65 & -9.14 & -53.0 & -124.7 & -146.1 & \\
-2.03 & & & -2.03 & -28.5 & -165.6 & -389.4 & -456.1 \\
\hline
 & 1 & +14.06 & +81.59 & +191.8 & +224.7 & -3.5 & -20.6
\end{array}$$

$$Q(x) = x^4 + 14.06x^3 + 81.59x^2 + 191.8x + 224.7$$

$$rx + s = -3.5x - 20.6$$

$$\begin{array}{r|rrrrr}
1 & 1 & +14.06 & +81.59 & +191.8 & +224.7 \\
-.65 & & -.65 & -8.72 & -46.05 & \\
-2.03 & & & -2.03 & -27.22 & -143.8 \\
\hline
 & 1 & +13.41 & +70.84 & +118.53 & +80.9
\end{array}$$

Therefore $\frac{\partial r}{\partial q} = -118.53$

$\frac{\partial s}{\partial q} = -80.9$

To find the partial derivatives with respect to p, divide $xQ(x)$ by $x^2 + .65x + 2.03$. Note that most of the work is identical with that in the preceding division.

$$\begin{array}{r|rrrrrr}
1 & 1 & +14.06 & +81.59 & +191.8 & +224.7 & +0 \\
-.65 & & -.65 & -8.72 & -46.05 & -77.0 & \\
-2.03 & & & -2.03 & -27.22 & -143.8 & -240.6 \\
\hline
 & 1 & +13.41 & +70.84 & +118.53 & +3.9 & -240.6
\end{array}$$

Then $\frac{\partial r}{\partial p} = -3.9$, $\frac{\partial s}{\partial p} = +240.6$

Equations (3) become

$$(4)\quad \begin{aligned} -3.9\Delta p - 118.53\Delta q &= 3.5 \\ 240.6\Delta p - 80.9\Delta q &= +20.6 \end{aligned}$$

Their solution is

$$\Delta p = +.075, \quad \Delta q = -.032$$

so that the next approximation to the quadratic factor is $x^2 + .725x + 1.98$.

One more step:

$$\begin{array}{r|rrrrrrr} 1 & 1 & +14.71 & +92.76 & +273.3 & +515 & +532 & +435.5 \\ -.725 & & -.725 & -10.14 & -58.46 & -135.68 & -159.25 & \\ -1.98 & & & -1.98 & -27.69 & -159.67 & -370.56 & -434.91 \\ \hline & 1 & +13.985 & -80.64 & +187.15 & +219.65 & +2.19 & +0.59 \end{array}$$

$$r = 2.19 \qquad s = 0.59$$

$$\begin{array}{r|rrrrrr} 1 & 1 & +13.985 & +80.64 & +187.15 & +219.65 & +[0] \\ -.725 & & -.725 & -9.61 & -50.06 & -[80.36] & \\ -1.98 & & & -1.98 & -26.25 & -136.72 & -[219.46] \\ \hline & 1 & +13.26 & +69.05 & +110.84 & \begin{array}{r} +82.93 \\ +[2.57] \end{array} & -[219.46]^{*} \end{array}$$

Equations (3) become:

$$-2.57\Delta p - 110.84\Delta q = -2.19$$

$$219.46\Delta p - 82.93\Delta q = -0.59$$

Their solution is

$$\Delta p = .0047, \quad \Delta q = .0196$$

and therefore the factor is $x^2 + .7297x + 1.996$. The other factor is $x^4 + 13.98x^3 + 80.56x^2 + 186.6x + 218$. This can be factored in the same way as the original polynomial.

Additional Notes

1. If all the roots of the polynomial equation have their real parts negative (this is the most common case in practice), then a better first approximation can be found

---

* In order to combine the division of Q(x) and xQ(x) into one operation, the additional numbers needed are put into brackets.

by using an idea of Lin in the paper previously mentioned.

Let

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$

Use as a first approximation

$$a_{n-2}x^2 + a_{n-1}x + a_n = a_{n-2}(x^2 + \frac{a_{n-1}}{a_{n-2}} x + \frac{a_n}{a_{n-2}})$$

2. Method I will converge* to the quadratic factor whose roots are smallest in absolute value. For an equation of odd degree with one real root smaller in absolute value than all the other complex roots, it is necessary to remove the real root before using Méthod I.

Another possibility is to find the quadratic factor whose roots are largest in absolute value. This can be done by assuming a trial factor and then dividing in back first, that is, divide in ascending powers and then in descending powers, instead of dividing in descending powers and then in ascending.

3. At each step in Method I the difference between the coefficients of the exact factor and the coefficients of the trial factor is approximately k times the corresponding difference at the preceding step. k depends directly on the ratio of the absolute value of the smallest pair of roots to the absolute value of the other roots.

At each step in Method II the coefficient difference is approximately the square of the corresponding difference

---

* A proof of convergents will be given in a mathematical journal at some future date.

at the preceding step. Therefore, if the first approximation is close enough, Method II will converge much more quickly than I.

In practice, if more than 5 per cent accuracy is needed in the roots, Method II should be used after four or five trials of Method I.

4. Both methods can be applied to find real factors of any degree. As an example we shall find a cubic factor of $x^5 + 17x^4 + 103x^3 + 297x^2 + 422x + 240$.

For a first approximation take

$103x^3 + 297x^2 + 422x + 240$ or $x^3 + \frac{297}{103} x^2 + \frac{422}{103} x + \frac{240}{103}$.

The first step in Method I can be set up as follows:

1	1	+ 17	+ 103	103	+ 297	+ 422	+ 240	1
-2.9		- 2.9	- 40.9	- 49.7	- 88.4	- 58.3		- .243
-4.2			- 4.2	- 6.2	- 4.1			- .017
-2.4								
	1	+ 14.1	+ 57.9	47.1	+ 204.5	+ 363.7	+ 240	

The second approximation is

$47.1 \left(x^3 + \frac{204.5}{47.1} x^2 + \frac{363.7}{47.1} x + \frac{240}{47.1}\right)$.

After five steps the result is

$(x^3 + 5.5x^2 + 11.3x + 8.3)\ (x^2 + 11.5x + 28.5)$.

NYU
AMG-
53 Friedman
c.1

Complex roots of pol
equati

NYU
AMG-
53 Friedman
c.1

AUTHOR

Complex roots of polynomial

TITLE

equations

DATE DUE	BORROWER'S NAME	ROOM NUMBER
JUN 8 '64	E. [illegible]	
	W. C. ~~[illegible]~~	
MAR 23 '65		
	~~H. [illegible]~~	

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.

Date Due

JUN 8 '64			
MAR 23 '65			

PRINTED IN U. S. A.